

NONSYMMETRIC MACDONALD POLYNOMIALS, DEMAZURE MODULES AND PBW FILTRATION

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ABSTRACT. The Cherednik-Orr conjecture expresses the $t \rightarrow \infty$ limit of the nonsymmetric Macdonald polynomials in terms of the PBW twisted characters of the affine level one Demazure modules. We prove this conjecture in several special cases.

INTRODUCTION

The Macdonald symmetric functions $P_\lambda(x, q, t)$ [M1] form a remarkable class of polynomials. These polynomials depend on the variables $x = (x_1, \dots, x_n)$ and two parameters q and t . The Macdonald symmetric functions can be specialized to the Hall-Littlewood polynomials ($q = 0$), to Schur polynomials ($q = t = 0$) and to Jack symmetric polynomials ($q = t^\alpha, t \rightarrow 1$).

The polynomials $P_\lambda(x, q, t)$ have a nonsymmetric version $E_\lambda(x, q, t)$ (see [Ch1], [O], [M2]). The symmetric functions $P_\lambda(x, q, t)$ can be reconstructed from $E_\lambda(x, q, t)$ via certain symmetrization over the Weyl group. The nonsymmetric Macdonald polynomials have many nice and interesting properties. In particular, they are known to be related to the representation theory of the affine Lie algebras (see [S], [I]). More precisely, the $t \rightarrow 0$ limit $E_\lambda(x, q, 0)$ coincides with the character of the corresponding level one Demazure module. In the recent papers [CO1], [CO2], [OS], [CF] the $t \rightarrow \infty$ limit of the nonsymmetric Macdonald polynomials was studied. In particular, it was shown that $E_\lambda(x, q, \infty)$ are polynomials in x and q^{-1} . Moreover, these polynomials have non-negative coefficients. Thus it is natural to expect a relation with the representation theory.

Let \mathfrak{g} be a simple Lie algebra and λ be an anti-dominant weight. We denote by W_λ be the corresponding level one Demazure module with the extremal vector w_λ . All Demazure modules are invariant with respect to the energy operator d from the affine Kac-Moody algebra. An important special property of the Demazure modules with antidominant highest weight is their invariance with respect to $\mathfrak{g} = \mathfrak{g} \otimes 1$. We assume that $dw_\lambda = 0$ and thus the eigenvalues of d on W_λ are nonnegative. We have the Kac-Moody character

$$\text{ch}_{KM} W_\lambda = \sum_{k \geq 0} q^k \text{ch}\{v \in W_\lambda, dv = kv\},$$

wher ch denotes the usual character with respect to the Cartan subalgebra of \mathfrak{g} . One knows that $\text{ch}_{KM} W_\lambda = E_\lambda(x, q, 0)$ (see [S]).

Cherednik and Orr conjectured that $E_\lambda(x, q, \infty)$ coincides with the PBW twisted character of W_λ . To give the precise formulation of the conjecture, recall that the Demazure modules W_λ are cyclic modules for the current algebra $\mathfrak{g} \otimes \mathbb{C}[t]$ with w_λ being the cyclic vector. The PBW filtration on the universal enveloping algebra of the current algebra induces the increasing filtration F_s on the Demazure module. Each space of this filtration is invariant with respect to the Cartan subalgebra and the associated graded space W_λ^{gr} is bi-graded by the operators d and by the PBW-grading operator D . We have the PBW character

$$\text{ch}_{PBW} W_\lambda = \sum_{k,s \geq 0} q^k p^s \text{ch} \{v \in F_s / F_{s-1}, dv = kv\}.$$

Cherednik and Orr put forward the following conjecture [CO1], Conjecture 2.7:

Conjecture 0.1. Assume that λ is an antidominant weight. Then

$$E_\lambda(x, q^{-1}, \infty) = \text{ch}_{PBW} W_\lambda|_{p=q}.$$

Several checks on the level of representations of finite-dimensional algebra were worked out in [CF]. The goal of this paper is to prove the conjecture in type A in several cases. Namely, we prove the following theorem:

Theorem 0.2. *Let \mathfrak{g} be of type A . Then the Cherednik-Orr conjecture is true if the dual of λ is equal to a multiple of a fundamental weight or to a linear combination of the first and the last fundamental weights.*

The paper is organized in the following way. In section 1 we collect main definitions and constructions about Demazure modules and PBW filtration. We also derive PBW bases for special Demazure modules. In section 2 we recall the Haglund-Haiman-Loehr formula [HHL] for the nonsymmetric Macdonald polynomials, derive the explicit combinatorial description of the $t \rightarrow \infty$ limit and study the properties of the polynomials $E_\lambda(x, q^{-1}, \infty)$. Finally, in section 3, we prove Theorem 0.2.

1. DEMAZURE MODULES AND PBW FILTRATION

1.1. Demazure modules and PBW filtration. Let us briefly recall the main ingredients (see [Kac], [Kum] for more details).

Let \mathfrak{g} be a simple Lie algebra. We fix a Cartan decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h}$, $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{h}$. Let Δ_+ be the set of positive roots of \mathfrak{g} , n be the rank of \mathfrak{g} and let $\alpha_i \in \Delta_+$, $i = 1, \dots, n$ be the set of simple roots. We denote by ω_i , $i = 1, \dots, n$ the fundamental weights. Let $P = \bigoplus_{i=1}^n \mathbb{Z}\omega_i$ be the weight lattice and let $P_+ = \bigoplus_{i=1}^n \mathbb{Z}_{\geq 0}\omega_i$ be the subset of dominant integral weights. Let $\lambda \in P_+$; we denote by V_λ the irreducible \mathfrak{g} -module with highest weight λ . For $\alpha \in \Delta_+$, let $f_\alpha \in \mathfrak{n}^-$ and $e_\alpha \in \mathfrak{n}$ be the corresponding Chevalley generators.

For a Lie algebra \mathfrak{a} we denote by $\mathfrak{a}[t] = \mathfrak{a} \otimes \mathbb{C}[t]$ the corresponding current algebra. We set $a[k] = a \otimes t^k \in \mathfrak{a}[t]$, $a \in \mathfrak{a}$, $k \geq 0$.

Let $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}K \oplus \mathbb{C}d$ be the affine Lie algebra; in particular, K is central and $[d, a \otimes t^k] = -ka \otimes t^k$. The current algebra $\mathfrak{g}[t]$ is naturally a subalgebra of $\widehat{\mathfrak{g}}$. We have the Cartan decomposition

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{n}^-} \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}.$$

For example, $\widehat{\mathfrak{n}} = \mathfrak{g} \otimes t\mathbb{C}[t] \oplus \mathfrak{n} \otimes 1$, $\widehat{\mathfrak{h}} = \mathfrak{h} \otimes 1 \oplus \mathbb{C}K \oplus \mathbb{C}d$.

Let $L = L_{\lambda,k}$ be an integrable irreducible highest weight $\widehat{\mathfrak{g}}$ module with the highest weight vector $v_{\lambda,k}$. The element K acts on L as the scalar k and this scalar is called the level of L . The highest weight of $L_{\lambda,k}$ is the pair (λ, k) , where $\lambda \in P_+$ and k is the level of $L_{\lambda,k}$. We have the condition $(\lambda, \theta) \leq k$, where θ is the highest root of \mathfrak{g} .

Remark 1.1. To make the pair (λ, k) into an honest weight of $\widehat{\mathfrak{g}}$ one has to specify the eigenvalue of the energy operator d on $v_{\lambda,k}$. However, this value is not important since the action of d on a $\widehat{\mathfrak{g}}$ module can be shifted by an arbitrary scalar. We choose the convenient shift depending on a concrete situation.

Let W be the finite Weyl group of \mathfrak{g} with the longest element w_0 . For $\lambda \in P$ we denote the dual weight $w_0\lambda$ by λ^* . In particular, if $\lambda \in P_+$, then λ^* is the lowest weight of the irreducible \mathfrak{g} module V_λ . Let \widehat{W} be the corresponding affine Weyl group; thus, \widehat{W} is the semi-direct product of W with the root lattice. The finite Weyl group naturally acts on the space of weights of \mathfrak{g} and \widehat{W} acts on the space of affine weights. For any integrable weight (λ, k) and $w \in \widehat{W}$ the weight space of the weight $w(\lambda, k)$ is one-dimensional. We fix one vector in each corresponding space and denote it by $v_{w(\lambda,k)}$. The Demazure module $D_w(\lambda)$ is defined as $D_w(\lambda) = U(\widehat{\mathfrak{n}})v_{w(\lambda,k)}$. We note that $D_w(\lambda)$ is not always invariant with respect to the action of $\mathfrak{g} = \mathfrak{g} \otimes 1$.

In what follows we only consider the level one modules. In this case for any weight $\mu \in P$ there exists unique integrable weight $(\lambda, 1)$ and $w \in \widehat{W}$ such that $w(\lambda, 1) = (\mu, 1)$. If μ is antidominant, i.e. $w_0\mu \in P_+$, then the Demazure module $D_w(\lambda)$ is $\mathfrak{g} \otimes 1$ -invariant. Assume that $w(\lambda, 1) = (\mu, 1)$. We use the shorthand notation $D_w(\mu) = W_\mu$, $v_{w(\lambda,1)} = w_\mu$. In particular, one has $U(\mathfrak{n} \otimes 1)w_\mu \simeq V_{\mu^*}$ with w_μ being the lowest weight vector. One also has $W_\mu = U(\mathfrak{n}[t])w_\mu$. The modules W_μ play important role in representation theory and in the theory of Macdonald polynomials (see e.g. [CL], [FL1], [FL2], [Kn], [S], [I]). In particular, W_μ are Weyl modules and fusion modules for antidominant μ .

Fixing $dw_\mu = 0$, we obtain the Kac-Moody character of W_μ , which is a polynomial in q :

$$\text{ch}_{KM} W_\mu = \sum_{r \geq 0} q^r \text{ch} \{w \in W_\mu : dw = rw\},$$

where ch is the \mathfrak{h} -character. In particular, $\text{ch}_{KM} W_\mu|_{q=0} = \text{ch} V_{\mu^*}$.

Let $U(\mathfrak{n}[t])_s$ be the PBW filtration on the universal enveloping algebra. Since $W_\mu = U(\mathfrak{n}[t])w_\mu$ we obtain the induced filtration on the Demazure module. Let W_μ^{gr} be the associated graded module; thus

$$W_\mu^{gr} = \bigoplus_{s \geq 0} W_\mu^{gr}(s), \quad W_\mu^{gr}(s) = \frac{U(\mathfrak{n}[t])_s w_\mu}{U(\mathfrak{n}[t])_{s-1} w_\mu}.$$

We note that W_μ^{gr} is a representation of the abelian Lie algebra $\mathfrak{n}^a[t]$, where \mathfrak{n}^a is the abelian Lie algebra with the underlying vector space \mathfrak{n} . Let D be the PBW-degree operator on W_μ^{gr} , i.e. $D|_{W_\mu^{gr}(s)} = s \cdot \text{Id}$. Let $W_\mu^{gr}(s, r)$ be the set of vectors $v \in W_\mu^{gr}(s)$ such that $dv = rv$. We note that each $W_\mu(s, r)$ is naturally an \mathfrak{h} module. We define the PBW character of W_μ as

$$\text{ch}_{PBW} W_\mu^{gr} = \sum_{r, s \geq 0} q^r p^s \text{ch} W_\mu^{gr}(s, r).$$

Remark 1.2. The computation of the PBW character of W_μ^{gr} looks very interesting, but is out of reach at the moment even in type A . One possible way to solve the problem is to find a basis of W_μ^{gr} , i.e. a basis of W_μ compatible with the PBW filtration (see [FFL1], [FFL2], [FFL3], [G] for the PBW bases of V_λ). Below we describe the PBW bases for two special classes of Demazure modules.

1.2. PBW basis. Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$. Let $\alpha_{i,j} = \alpha_i + \dots + \alpha_j$ ($1 \leq i \leq j \leq n$) be the set of positive roots. We denote by $f_{i,j} = f_{\alpha_{i,j}}$, $e_{i,j} = e_{\alpha_{i,j}}$ the Chevalley generators of \mathfrak{g} . Let $f_{i,j}[k] = f_{i,j} \otimes t^k$, $e_{i,j}[k] = e_{i,j} \otimes t^k$. We list some properties of the Demazure modules W_λ in the following lemma.

Lemma 1.3. *Let $\lambda^* = \sum_{i=1}^n m_i \omega_i \in P_+$. Then*

- W_λ is generated from the cyclic vector $w_\lambda \in W_\lambda$ by the action of the operators $e_\alpha[k] = e_\alpha \otimes t^k$, $\alpha \in P_+$ and $k \geq 0$.
- $\dim W_\lambda = \prod_{i=1}^n (\dim V_{\omega_i})^{m_i}$, $W_\lambda \simeq \bigotimes_{i=1}^n V_{\omega_i}^{\otimes m_i}$ as \mathfrak{g} -modules.
- W_λ is a $\mathfrak{g} \otimes \mathbb{C}[t]$ -module; it is isomorphic to the Weyl module.
- $e_\alpha[k]w_\lambda = 0$ for $k \geq \sum_i m_i$.

In what follows it will be convenient to use the \mathfrak{gl}_n notation for the characters of V_λ and of W_λ : we represent the characters as functions in variables x_1, \dots, x_{n+1} . For example, the character of the lowest weight vector $w_{\omega_r^*} \in W_{\omega_r^*}$ is equal to $x_{n+1} \dots x_{n-r+2}$. In general, if $\lambda^* = \sum m_i \omega_i$, then

$$\text{ch} w_\lambda = x_{n+1}^{m_1} (x_{n+1} x_n)^{m_2} \dots (x_{n+1} \dots x_2)^{m_n} = \prod_{i=2}^{n+1} x_i^{\lambda_i},$$

where $\lambda_i = m_n + m_{n-1} + \dots + m_{n-i+2}$.

Proposition 1.4. *Let $\lambda^* = m\omega_r$. Then one has:*

- The module W_λ is generated from the vector w_λ by the action of the polynomial algebra on variables $e_{i,j}[k]$, $i \leq n - r + 1 \leq j$, $k \geq 0$.
- The PBW degree of a monomial $e_{i_1, j_1}[k_1] \dots e_{i_c, j_c}[k_c]$ is equal to c .

- The PBW character and the Kac-Moody character are related by the formula

$$\text{ch}_{PBW} W_\lambda^{gr}(x_1, \dots, x_{n+1}, p, q) = \text{ch}_{KM} W_\lambda(px_1, \dots, px_{n+1-r}, x_{n+2-r}, \dots, x_{n+1}, q).$$

Proof. The first claim is an immediate consequence of $e_\alpha w_\lambda = 0$ if $(\lambda, \alpha) = 0$. To prove the second statement we note that the PBW degree of a vector $v \in W_\lambda$ is equal to the coefficient of α_{n+1-r} in the difference between the weight of v and that of w_λ (in each $e_{i,j}$, $i \leq n-r \leq j$ the simple root α_{n-r+1} shows up exactly once). The last claim follows from the observation that the character of $e_{i,j}$ is equal to $x_i x_{j+1}^{-1}$. \square

Remark 1.5. Proposition 1.4 implies that any basis for the Weyl module $W_{\omega_r^*}$ is the PBW basis.

In the rest of the section we describe the PBW basis in the case $\lambda^* = m_1 \omega_1 + m_2 \omega_n$. We follow the notation from [CL]. Let $l \geq 0$ and let $\mathbf{s} = (\mathbf{s}(1) \leq \dots \leq \mathbf{s}(l))$ be a collection of nonnegative integers. For a positive root α we use the notation

$$e_\alpha(l, \mathbf{s}) = \prod_{1 \leq k \leq l} e_\alpha[s(k)].$$

If $\alpha = \alpha_{i,j}$, we abbreviate $e_{\alpha_{i,j}}(l, \mathbf{s})$ by $e_{i,j}(l, \mathbf{s})$. We first recall several lemmas from [CL].

Let $\mathfrak{g} = \mathfrak{sl}_2$.

Lemma 1.6. *The vectors $e_{1,1}(l, \mathbf{s}) w_{m\omega_1}$ subject to the condition $\mathbf{s}(l) \leq m-l$ form a basis of $W_{(m\omega_1)^*}$. The defining relations of the $\mathfrak{sl}_2[t]$ -module $W_{m\omega_1}$ are $f[k] w_{(m\omega_1)^*}$ ($k \geq 0$), $h[k] w_{m\omega_1}$ ($k > 0$), $e[0]^N$ ($N > m$).*

Lemma 1.7. *Let $\lambda = (m\omega_n)^*$. Then the vectors $\prod_{k=1}^n e_{1,k}(l_k, \mathbf{s}_k) w_\lambda$ subject to the conditions*

$$\mathbf{s}_1(l_1) \leq m - l_1, \mathbf{s}_2(l_2) \leq m - l_1 - l_2, \dots, \mathbf{s}_{n-1}(l_{n-1}) \leq m - l_1 - \dots - l_{n-1}$$

form a PBW basis of W_λ .

Let $\lambda = (m\omega_1)^$. Then the vectors $\prod_{k=1}^n e_{k,n}(l_k, \mathbf{s}_k) w_\lambda$ subject to the conditions*

$$\mathbf{s}_n(l_n) \leq m - l_n, \mathbf{s}_{n-1}(l_{n-1}) \leq m - l_n - l_{n-1}, \dots, \mathbf{s}_1(l_1) \leq m - l_n - \dots - l_1$$

form a PBW basis of W_λ .

Proof. It is proved in [CL] that the vectors described above form the PBW bases of $W_{(m\omega_n)^*}$ and of $W_{(m\omega_1)^*}$; these are PBW bases thanks to Proposition (1.4). \square

We prove the following theorem.

Theorem 1.8. *Let $\lambda^* = m_1\omega_1 + m_2\omega_n$. Then the module W_λ has a PBW basis of the form*

$$(1.1) \quad e_{1,n}(l_{1,n}, \mathbf{s}_{1,n}) \prod_{k=1}^{n-1} e_{1,k}(l_{1,k}, \mathbf{s}_{1,k}) \prod_{k=2}^n e_{k,n}(l_{k,n}, \mathbf{s}_{k,n}) w_\lambda$$

subject to the conditions

$$(1.2) \quad \mathbf{s}_{1,k}(l_{1,k}) \leq m_2 - l_{1,1} - \cdots - l_{1,k}, \quad k = 1, \dots, n-1,$$

$$(1.3) \quad \mathbf{s}_{k,n}(l_{k,n}) \leq m_1 - l_{n,n} - \cdots - l_{k,n}, \quad k = n, \dots, 2,$$

$$(1.4) \quad \mathbf{s}_{1,n}(l_{1,n}) \leq m_1 + m_2 - l_{1,1} - \cdots - l_{1,n-1} - l_{1,n} - l_{2,n} - \cdots - l_{n,n}.$$

Lemma 1.9. *The number of solutions of inequalities (1.2), (1.3), (1.4) is equal to the dimension of W_λ .*

Proof. We know that $\dim W_\lambda = (n+1)^{m_1+m_2}$. So we only need to show that

$$\sum_{\substack{l_{1,1}+\cdots+l_{1,n-1}\leq m_2 \\ l_{n,n}+\cdots+l_{n-1,n}\leq m_1}} 2^{m_1+m_2-l_{1,1}-\cdots-l_{1,n-1}-l_{n,n}-\cdots-l_{n-1,n}} \binom{m_2}{l_{1,1}, \dots, l_{1,n-1}} \binom{m_1}{l_{n,n}, \dots, l_{n-1,n}}$$

is equal to $(n+1)^{m_1+m_2}$, which is clear. \square

Let F_s be the PBW filtration on W_λ . For any $\alpha \in \Delta_+$ and $k \geq 0$ one has $f_\alpha[k]F_s \subset F_s$. Hence we obtain the induced PBW-degree zero operators on W_λ^{gr} , which we denote by $\partial_\alpha[k]$. Recall that W_λ^{gr} can be represented as the quotient of the polynomial ring in variables $e_\alpha[k]$. We have the following easy lemma:

Lemma 1.10. *The operators $\partial_\alpha[k]$ are induced by the differential operators (which we also denote by $\partial_\alpha[k]$) on the polynomial algebra in variables $e_\alpha[k]$. One has $\partial_\alpha[k]e_\beta[r] = 0$ unless $[f_\alpha, e_\beta] = c_{\alpha,\beta}^\gamma e_\gamma$ for some $\gamma \in \Delta_+$. If this equality holds, then $\partial_\alpha[k]e_\beta[r] = c_{\alpha,\beta}^\gamma e_\gamma[k+r]$.*

Now let θ be the highest root of \mathfrak{g} . Let \mathfrak{sl}_2^θ be the \mathfrak{sl}_2 algebra generated by f_θ and e_θ .

Lemma 1.11. *The differential operators $\partial_\theta[k]$ vanish. The operators $f_\theta[k]$ map F_s to F_{s-1} and hence induce the degree minus one operators $f_\theta^{gr}[k]$ on W_λ^{gr} . The operators $f_\theta^{gr}[k]$, $e_\theta[k]$ and $h_\theta[k]$ form the Lie algebra $\mathfrak{sl}_2^\theta[t]$, isomorphic to $\mathfrak{sl}_2[t]$.*

Proof. We note that $[f_\theta, e_\alpha] \in \mathfrak{b}$ for any $\alpha \in \Delta_+$ and hence $f_\theta[k]F_s \subset F_{s-1}$. Therefore we obtain that the operators $\partial_\theta[k]$ vanish, but there exists degree minus one operators $f_\theta^{gr}[k]$ on W_λ^{gr} . The last statement is clear. \square

We now prove the main theorem. We first sketch the proof in the \mathfrak{sl}_3 -case, and then give the proof for general n .

Lemma 1.12. *Let $\mathfrak{g} = \mathfrak{sl}_3$, $\lambda^* = m_1\omega_1 + m_2\omega_2$. Then the vectors*

$$(1.5) \quad e_{1,1}(l_{1,1}, \mathbf{s}_{1,1})e_{2,2}(l_{2,2}, \mathbf{s}_{2,2})e_{1,2}(l_{1,2}, \mathbf{s}_{1,2})w_\lambda$$

subject to the conditions

$$(1.6)$$

$$s_{1,1}(l_{1,1}) \leq m_2 - l_{1,1}, \quad s_{2,2}(l_{2,2}) \leq m_1 - l_{2,2}, \quad s_{1,2}(l_{1,2}) \leq m_1 + m_2 - l_{1,1} - l_{2,2} - l_{1,2}$$

form a basis of W_λ^{gr} .

Proof. Consider an arbitrary vector of the form (1.5). We want to show that it can be rewritten as a linear combination of monomials subject to conditions (1.6). Restricting the module W_λ to the subalgebras $\mathfrak{sl}_2[t]$ corresponding to simple roots we can assume that

$$s_{1,1}(l_{1,1}) \leq m_2 - l_{1,1}, \quad s_{2,2}(l_{2,2}) \leq m_1 - l_{2,2}$$

(since we know that these restrictions produce basis in the \mathfrak{sl}_2 case, see Lemma 1.6). Now we note that

$$(1.7) \quad e_{1,2}[0]^m e_{1,1}(l_{1,1}, \mathbf{s}_{1,1})e_{2,2}(l_{2,2}, \mathbf{s}_{2,2})w_\lambda = 0$$

provided $m + l_{1,1} + l_{2,2} > m_1 + m_2$ (this can be shown via applying the differential operators, see the proof in the general case below). Now consider the action of the algebra $\mathfrak{sl}_2^\theta[t]$. We note that

$$f_{1,2}[k]e_{1,1}(l_{1,1}, \mathbf{s}_{1,1})e_{2,2}(l_{2,2}, \mathbf{s}_{2,2})w_\lambda = 0.$$

Indeed, $[f_{1,2}, e_{1,1}] = f_{2,2}$ and $[f_{1,2}, e_{2,2}] = -f_{1,1}$. Therefore, the PBW degree of $f_{1,2}[k]e_{1,1}(l_{1,1}, \mathbf{s}_{1,1})e_{2,2}(l_{2,2}, \mathbf{s}_{2,2})w_\lambda$ is at most $l_{1,1} + l_{2,2} - 2$. Now using (1.7) and the action $\mathfrak{sl}_2[t]$ we obtain the desired claim thanks to Lemma 1.6. \square

Now we give the general proof.

Proof. Because of Lemma 1.9 it suffices to prove that any vector in W_λ can be written as a linear combination of vectors (1.1) subject to the conditions (1.2), (1.3), (1.4). First, let us restrict W_λ to the subalgebra $\mathfrak{sl}_{n-1}[t]$, corresponding to simple roots $\alpha_1, \dots, \alpha_{n-1}$. Then we have all the relations from $W_{(m_2\omega_n)^*}$ and hence we can assume that all the conditions (1.2) hold. Similarly we can assume that restrictions (1.3) are satisfied.

We have

$$(1.8) \quad e_{1,n}[0]^m \prod_{j=1}^{n-1} e(l_{1,j}, \mathbf{s}_{1,j}) \prod_{i=2}^n e(l_{i,n}, \mathbf{s}_{i,n})w_\lambda = 0$$

provided $m + \sum_{j=1}^{n-1} l_{1,j} + \sum_{i=2}^n l_{i,n} > m_1 + m_2$. In fact, we know that $e_{1,n}[0]^m w_\lambda = 0$ if $m > m_1 + m_2$; in addition

$$\begin{aligned} \partial_{1,i-1}[k]^r e_{1,n}[0]^m &= \text{const. } e_{i,n}[k]^r e_{1,n}[0]^{m-r}, \\ \partial_{j+1,n}[k]^r e_{1,n}[0]^m &= \text{const. } e_{1,j}[k]^r e_{1,n}[0]^{m-r} \end{aligned}$$

which proves (1.8). The difference with the \mathfrak{sl}_3 case is that $f_\theta^{gr}[k]$ does not always kill $\prod_{j=1}^{n-1} e(l_{1,j}, \mathbf{s}_{1,j}) \prod_{i=2}^n e(l_{i,n}, \mathbf{s}_{i,n}) w_\lambda$. To resolve this difficulty we introduce a filtration G_\bullet on W_λ^{gr} . The spaces G_ν of the filtration are labeled by the weights $\nu \in P$; G_ν is spanned by the vectors of the form (1.1) subject to conditions (1.2), (1.3) (but not (1.4)) such that the weight of

$$\prod_{k=1}^{n-1} e_{1,k}(l_{1,k}, \mathbf{s}_{1,k}) \prod_{k=2}^n e_{k,n}(l_{k,n}, \mathbf{s}_{k,n}) w_\lambda$$

is at most $\lambda - \nu$. Then in the associated graded module we have the action of the algebra $\mathfrak{sl}_2^\theta[t]$ and the argument of Lemma 1.12 applies. \square

2. NONSYMMETRIC MACDONALD POLYNOMIALS

2.1. The Haglund-Haiman-Loehr formula. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be an n -tuple of integers. The type A nonsymmetric Macdonald polynomials $E_\lambda(x, q, t)$ are polynomials in variables $x = (x_1, \dots, x_n)$ with coefficients in $\mathbb{Q}(q, t)$. They are simultaneous eigenfunctions of the Cherednik operators (see e.g. [HHL]). In what follows we need the following Knop-Sahi property of the Macdonald polynomials. Let

$$(2.1) \quad \pi(\lambda_1, \dots, \lambda_n) = (\lambda_n + 1, \lambda_1, \dots, \lambda_{n-1}),$$

$$(2.2) \quad (\Psi f)(x_1, \dots, x_n) = x_1 f(x_2, \dots, x_n, q^{-1}x_1).$$

Then $E_{\pi(\lambda)}(x, q, t) = q^{\lambda_n} \Psi E_\lambda(x, q, t)$.

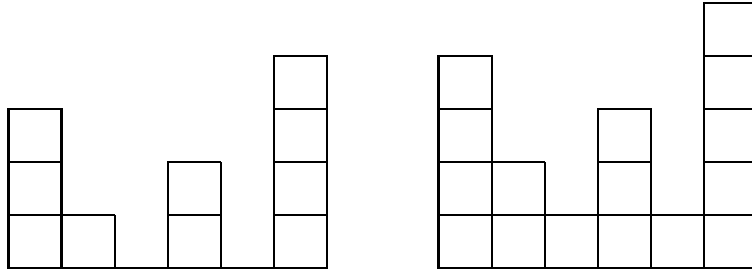
We use the explicit combinatorial description of the nonsymmetric Macdonald polynomials from [HHL]. For a composition $\lambda = (\lambda_1, \dots, \lambda_n)$ the column diagram $dg'(\lambda)$ is the set

$$dg'(\lambda) = \{(i, j) \in \mathbb{N}^2 : 1 \leq i \leq n, 1 \leq j \leq \lambda_i\}$$

The augmented diagram $\widehat{dg}(\lambda)$ is defined by

$$\widehat{dg}(\lambda) = dg'(\lambda) \cup \{(i, 0) : 1 \leq i \leq n\},$$

i.e. one box is added to the bottom of each column. For example, for the composition $\lambda = (3, 1, 0, 2, 0, 4)$ one has the following diagrams for $dg'(\lambda)$ and $\widehat{dg}(\lambda)$:



In what follows we will be mostly interested in anti-dominant diagrams, i.e. such that $\lambda_i \leq \lambda_j$, if $i < j$.

A filling of λ is the map $\sigma : dg'(\lambda) \rightarrow \{1, \dots, n\}$. The associated augmented filling $\widehat{\sigma} : \widehat{dg}(\lambda) \rightarrow \{1, \dots, n\}$ agrees with σ on $dg'(\lambda)$ and $\widehat{\sigma}((j, 0)) = j$ for $j = 1, \dots, n$.

Two boxes are called attacking if they are in the same row or they are in consecutive rows, and the box in the lower row is to the right of the one in the upper row, i.e., they have the form $(i, j), (i_1, j-1)$ with $i < i_1$. A filling is called non-attacking, if there are no equal elements in attacking boxes.

For a box u let $d(u)$ be the box strictly below u . The descent of a filling σ is the set of boxes u such that $\widehat{\sigma}(u) > \widehat{\sigma}(d(u))$.

For a box $u = (i, j)$ let $leg(u)$ be the set of cells above u and $l(u) = |leg(u)| = \lambda_i - j$. We will also need the value $a(u)$ counting the cardinality of the arms of u . Define:

$$arm^{left}(u) = \{(i', j) \in dg'(\lambda) | i' < i, \lambda_{i'} \leq \lambda_i\}.$$

$$arm^{right}(u) = \{(i', j-1) \in \widehat{dg}(\lambda) | i' > i, \lambda_{i'} < \lambda_i\}.$$

$$arm(u) = arm^{left}(u) \cup arm^{right}(u)$$

Then $a(u) = |\{arm(u)\}|$.

Note that for anti-dominant diagrams we have:

$$arm(u) = arm^{left}(u) = \{(i', j) \in dg'(\lambda) | i' < i\}.$$

Let $Des(\widehat{\sigma})$ be the set of descents of $\widehat{\sigma}$ and

$$maj(\widehat{\sigma}) = \sum_{u \in Des(\widehat{\sigma})} (l(u) + 1).$$

A pair of attacking elements $u = (i, j)$ and $u' = (i', j')$ of $\widehat{\sigma}$ is an inversion if $\widehat{\sigma}(u) < \widehat{\sigma}(u')$ and $j = j', i < i'$ or $j+1 = j', i > i'$. Let $Inv(\widehat{\sigma})$ be the set of inversions of $\widehat{\sigma}$. Let

(2.3)

$$coinv(\widehat{\sigma}) = \sum_{u \in dg'(\lambda)} a(u) - |Inv(\widehat{\sigma})| + |\{(i < j : \lambda_i \leq \lambda_j)\}| + \sum_{u \in Des(\widehat{\sigma})} a(u).$$

For a filling σ of the diagram $dg'(\lambda)$ we define

$$x^\sigma = \prod_{i=1}^n x_i^{|\{u \in dg'(\lambda) : \sigma(u)=i\}|}.$$

Theorem 2.1. (*Haglund, Haiman, and Loehr*)

(2.4)

$$E_\lambda(x; q, t) = \sum_{\sigma \text{ non-attacking}} x^\sigma q^{maj(\widehat{\sigma})} t^{coinv(\widehat{\sigma})} \prod_{\substack{u \in dg'(\lambda) \\ \widehat{\sigma}(u) \neq \widehat{\sigma}(d(u))}} \frac{1-t}{1-q^{l(u)+1} t^{a(u)+1}}.$$

In our paper we only need the special compositions μ of the form $\mu = \pi^r(\lambda)$, where $\lambda = (\lambda_1 \leq \dots \leq \lambda_n)$ is anti-dominant and π^r is the power of the operator (2.1).

For an anti-dominant λ we have:

$$(2.5) \quad coinv(\widehat{\sigma}) = \sum_{u \in dg'(\lambda)} a(u) - |Inv(\widehat{\sigma})| + \frac{n(n-1)}{2} + \sum_{u \in Des(\widehat{\sigma})} a(u).$$

Note that we have $\frac{n(n-1)}{2}$ inversions in the bottom row of $\widehat{dg}(\lambda)$. Let $Inv'(\widehat{\sigma}) = Inv(\widehat{\sigma}) - \frac{n(n-1)}{2}$. We have

$$(2.6) \quad coinv(\widehat{\sigma}) = \sum_{u \in dg'(\lambda)} a(u) - |Inv'(\widehat{\sigma})| + \sum_{u \in Des(\widehat{\sigma})} a(u).$$

Remark 2.2. We note that for non-attacking σ there are no inversions in $\widehat{\sigma}$ between the cells in the first and zero rows.

For a composition $\mu = \pi^r(\lambda)$ with antidominant λ one has

$$|\{(i < j : \mu_i \leq \mu_j)\}| = \frac{r(r-1)}{2} + \frac{(n-r)(n-r-1)}{2}$$

and hence

$$(2.7) \quad coinv(\widehat{\sigma}) = \sum_{u \in dg'(\lambda)} a(u) - |Inv'(\widehat{\sigma})| - r(n-r) + \sum_{u \in Des(\widehat{\sigma})} a(u).$$

2.2. $t \rightarrow \infty$ limit. In this section we give a combinatorial formula for the $t \rightarrow \infty$ limit of the polynomials $E_\mu(x; q, t)$.

We note that $\lim_{t \rightarrow \infty} E_\mu(x; q, t) = \lim_{t \rightarrow \infty} \sum_{\sigma \text{ non-attacking}} x^\sigma t^{T(\sigma)} q^{Q(\sigma)}$, where

$$(2.8) \quad T(\sigma) = coinv(\widehat{\sigma}) - \sum_{\substack{u \in dg' \\ \widehat{\sigma}(u) \neq \widehat{\sigma}(d(u))}} a(u),$$

$$(2.9) \quad Q(\sigma) = maj(\widehat{\sigma}) - \sum_{\substack{u \in dg' \\ \widehat{\sigma}(u) \neq \widehat{\sigma}(d(u))}} (l(u) + 1) = - \sum_{\substack{u \in dg' \\ \widehat{\sigma}(u) < \widehat{\sigma}(d(u))}} (l(u) + 1).$$

For any cell $u = (i, j) \in \widehat{dg}(\lambda)$ we define the cell $\pi(u) \in dg'(\pi(\lambda))$ by

$$\pi(u) = \begin{cases} (i+1, j), & \text{if } i \neq n, \\ (1, j+1), & \text{if } i = n. \end{cases}$$

We note that $v \in arm(u)$ iff $\pi(v) \in arm(\pi(u))$ and $v \in leg(u)$ iff $\pi(v) \in leg(\pi(u))$; in particular, $a(u) = a(\pi(u))$ and $l(u) = l(\pi(u))$.

The following lemma (even for general μ) is known (see [CO1]). However we give a combinatorial proof below. The construction given in the proof will be very important later.

Lemma 2.3. *Let $\mu = \pi^r(\lambda)$ and assume that λ is antidominant. Then $\lim_{t \rightarrow \infty} E_\mu(x; q, t)$ is well defined polynomial in x and q^{-1} .*

Proof. We have to prove that for any non-attacking filling σ the power

$$(2.10) \quad \text{coinv}(\hat{\sigma}) - \sum_{\substack{u \in dg'(\mu) \\ \hat{\sigma}(u) \neq \hat{\sigma}(d(u))}} a(u)$$

is less than or equal to zero. Using 2.7 we have:

$$\begin{aligned} \text{coinv}(\hat{\sigma}) - \sum_{\substack{u \in dg'(\mu) \\ \hat{\sigma}(u) \neq \hat{\sigma}(d(u))}} a(u) = \\ \sum_{u \in dg'(\mu)} a(u) - |\text{Inv}'(\hat{\sigma})| - r(n-r) + \sum_{u \in \text{Des}(\hat{\sigma})} a(u) - \sum_{\substack{u \in dg'(\mu) \\ \hat{\sigma}(u) \neq \hat{\sigma}(d(u))}} a(u) = \\ \sum_{u \in dg'(\mu)} a(u) - |\text{Inv}'(\hat{\sigma})| - r(n-r) - \sum_{\substack{u \in dg'(\mu) \\ \hat{\sigma}(u) < \hat{\sigma}(d(u))}} a(u). \end{aligned}$$

We consider three cells $u = \pi^r(\tilde{u})$, $d(u)$ and $v \in \text{arm}(u)$ (i.e. $v = \pi^r(\tilde{v})$ for some $\tilde{v} \in \text{arm}(\tilde{u})$). Assume that $\hat{\sigma}(u) > \hat{\sigma}(d(u))$. Then we have $\hat{\sigma}(v) < \hat{\sigma}(u)$ or $\hat{\sigma}(v) > \hat{\sigma}(d(u))$. Hence if $\hat{\sigma}(u) > \hat{\sigma}(d(u))$ then for any element in $\text{arm}(u)$ we have at least one inversion. Let Inv^π be the set of inversions of the form $\pi^r(u), \pi^r(v)$. We have:

$$\begin{aligned} \sum_{u = \pi^r(\tilde{u})} a(u) - |\text{Inv}^\pi(\sigma)| - \sum_{\substack{u = \pi^r(\tilde{u}) \\ \hat{\sigma}(u) < \hat{\sigma}(d(u))}} a(u) \leq \\ \sum_{u = \pi^r(\tilde{u})} a(u) - \sum_{\substack{u = \pi^r(\tilde{u}) \\ \hat{\sigma}(u) < \hat{\sigma}(d(u))}} a(u) - \sum_{\substack{u = \pi^r(\tilde{u}) \\ \hat{\sigma}(u) > \hat{\sigma}(d(u))}} a(u) = 0, \end{aligned}$$

Elements in $dg'(\mu)$ that are not of the type $\pi^r(u)$ are $(1, 1), \dots, (r, 1)$. We note that if a filling is non-attacking then $\sigma(i, 1) = i, i = 1, \dots, r$. Hence these elements produce $\frac{r(r-1)}{2}$ inversions. Lengths of their arms are $n-1, \dots, n-r$. Summing up we obtain $\frac{r(r-1)}{2} + r(n-r)$. This completes the proof of Lemma. \square

Lemma 2.3 tells us that $T(\hat{\sigma})$ is less than or equal to zero. At the $t \rightarrow \infty$ limit only the fillings with $T(\hat{\sigma}) = 0$ survive. Any three elements considered in the previous Lemma give one negative summand to the power of t . Therefore if $\hat{\sigma}(u) \geq \hat{\sigma}(d(u))$ then for any element v in the arm of u only one of the inequalities $\hat{\sigma}(v) < \hat{\sigma}(u)$ and $\hat{\sigma}(v) > \hat{\sigma}(d(u))$ holds, $\hat{\sigma}(v)$ is not between $\hat{\sigma}(u)$ and $\hat{\sigma}(d(u))$. And if $\hat{\sigma}(u) < \hat{\sigma}(d(u))$ then for any v in the arm of u non of these equations hold, i. e. $\hat{\sigma}(v)$ is between $\hat{\sigma}(u)$ and $\hat{\sigma}(d(u))$. We thus obtain the following description of fillings such that the power of t (2.10) vanishes.

Proposition 2.4. *Let $\widehat{\sigma}$ be a non-attacking filling of the diagram $\widehat{dg}(\mu)$ such that $T(\widehat{\sigma}) = 0$. Let $u \in dg'(\mu)$, $\widehat{\sigma}(d(u)) = k$ and let $S = \{\sigma(v) | v \in \text{arm}(u)\} \cup \{\sigma(u)\}$. Then*

$$\sigma(u) = \begin{cases} \min_{x \in S, x \geq k} x, & \text{if } \exists x \in S, x \geq k, \\ \min_{x \in S} x, & \text{otherwise.} \end{cases}$$

Using the previous proposition we can obtain the following description of all fillings of $\widehat{dg}(\mu)$, $\mu = \pi^r(\lambda)$, such that $T(\widehat{\sigma}) = 0$. We denote by $dg'_j(\lambda)$ the j -th row of the diagram.

Lemma 2.5. *Assume that $T(\widehat{\sigma}) = 0$. Then the filling of $\pi^r(dg'_j(\lambda))$ and the set $S = \{\sigma(u), u \in \pi^r(dg'_{j+1}(\lambda))\}$ uniquely determines the filling of $\pi^r(dg'_{j+1}(\lambda))$. We fill $\pi^r(dg'_{j+1}(\lambda))$ cell by cell from $\pi^r(n, j+1)$ to $\pi^r(1, j+1)$. Given a cell $v \in \pi^r(dg'_{j+1}(\lambda))$, the value $\sigma(v)$ is determined as follows: let S' be the set of elements of S that are not used at the previous steps. Then $\sigma(v)$ is equal to:*

- (i) $\min\{a \in S', a \geq \sigma(d(v))\}$, if $\{a \in S', a \geq \sigma(d(v))\} \neq \emptyset$;
- (ii) $\min\{a \in S'\}$, if $\{a \in S', a \geq \sigma(d(v))\} = \emptyset$.

Proof. Immediate consequence of Proposition 2.4. □

In what follows we call the fillings $\sigma(\lambda)$ such that $T(\widehat{\sigma}) = 0$ *appropriate*.

Proposition 2.6. *Let $\lambda = (\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n)$ be an antidominant weight. Let $\widehat{\sigma}$ be a filling of the diagram $\widehat{dg}(\lambda)$ such that the elements of the lowest row are $\sigma(i, 0) = i$. Then there is only one appropriate filling with the fixed sets of elements in every row. For such a filling $\widehat{\sigma}$ one has*

$$Q(\sigma) = - \sum_{\widehat{\sigma}(u) < \widehat{\sigma}(d(u))} (l(u) + 1).$$

In particular, we obtain the following well-known corollary:

Corollary 2.7. *Let $m_i = \lambda_{n+1-i} - \lambda_{n-i}$. Then*

$$E_\lambda(x; 1, \infty) = \text{ch} \bigotimes_{i=1}^{n-1} V_{\omega_i}^{m_{i+1}-m_i},$$

where V_{ω_i} are fundamental representations.

Proof. An appropriate filling $\widehat{\sigma}$ contains a set of different elements in each row of $dg'(\lambda)$. Thanks to Proposition 2.6 we have a bijection between rows of our filling and elements of the standard basis of V_{ω_k} . The weight of the corresponding element of the basis equals to $\prod_{i=k+1}^n x^{\sigma(i,j)}$. □

2.3. Recurrent formula. Let λ be an antidominant composition such that $\lambda_1 = \dots = \lambda_{n-s} = 0 \neq \lambda_{n-s+1}$ (i.e. there are s cells in $dg'(\lambda)$). Let $\mathbf{a} = (a_{n-s+1}, \dots, a_n)$ be elements of the lowest row of an appropriate filling σ , i.e. the rule of Proposition 2.4 is satisfied (we assume that a_j lives in the j -th column). We denote the lowest row of σ by $\text{low}(\sigma)$ and define

$$k(\sigma) = (k_1, \dots, k_n), \quad k_i = |\{u | \sigma(u) = i\}|.$$

Recall that

$$E_\lambda(x, q^{-1}, \infty) = \sum_{\sigma: \text{appropriate}} q^{Q(\sigma)} x^{k(\sigma)}.$$

We introduce the notation which we extensively use below:

$$(2.11) \quad c_{\mathbf{a}}^\lambda(\mathbf{k}) = \sum_{\substack{\sigma: \text{low}(\sigma) = (a_1, \dots, a_s), \\ k(\sigma) = (k_1, \dots, k_n)}} q^{Q(\sigma)}.$$

Using Proposition 2.4 we have:

$$E_\lambda(x; q^{-1}, \infty) = \sum_{k_1, \dots, k_n \geq 0} c_{(n-s+1, \dots, n)}^{(0, \dots, 0, \lambda_{n-s+1}+1, \dots, \lambda_n+1)}(k_1, \dots, k_{n-s}, k_{n-s}+1, \dots, k_n+1) x_1^{k_1} \dots x_n^{k_n}.$$

Let s' be the number of cells in the lowest but one row. For a string $\mathbf{a} = (a_{n-s+1}, \dots, a_n)$ and a set B , $|B| = s' \leq s$, let $\mathbf{a}(B) = (b_{n-s'+1}, \dots, b_n)$ be an ordering of B obtained using the rule of Lemma 2.4.

Proposition 2.8. Let $\delta_1, \dots, \delta_n$ be the numbers defined by

$$\delta_i = \begin{cases} 1, & \text{if } i \in \{a_{n-s+1}, \dots, a_n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$c_{\mathbf{a}}^{(0, \dots, 0, \lambda_{n-s+1}+1, \dots, \lambda_n+1)}(k_1 + \delta_1, \dots, k_n + \delta_n) = \sum_{B: |B|=s'} c_{\mathbf{a}(B)}^\lambda(k_1, \dots, k_n) q^{\sum_{j: b_j < a_j} \lambda_j}.$$

Proof. This is an immediate consequence of Theorem 2.6. \square

Example 2.9. Consider the case $n = 3$ and partitions of the type $\lambda = (0, m_2, m_1 + m_2)$ (this is the general case for \mathfrak{sl}_3). Then we obtain the following equations:

$$\begin{aligned} c_{(2,1)}^{(0, m_2+1, m_1+m_2+1)}(k_1 + 1, k_2 + 1, k_3) &= c_{(2,1)}^{(0, m_2, m_1+m_2)}(k_1, k_2, k_3) + \\ &+ c_{(3,1)}^{(0, m_2, m_1+m_2)}(k_1, k_2, k_3) + c_{(3,2)}^{(0, m_2, m_1+m_2)}(k_1, k_2, k_3); \\ c_{(3,1)}^{(0, m_2+1, m_1+m_2+1)}(k_1 + 1, k_2, k_3 + 1) &= c_{(2,1)}^{(0, m_2, m_1+m_2)}(k_1, k_2, k_3) q^{m_2} + \\ &+ c_{(3,1)}^{(0, m_2, m_1+m_2)}(k_1, k_2, k_3) + c_{(3,2)}^{(0, m_2, m_1+m_2)}(k_1, k_2, k_3). \end{aligned}$$

Using these equations we obtain:

$$(2.12) \quad c_{(3,1)}^{(0,m_2+1,m_1+m_2+1)}(k_1+1, k_2, k_3+1) = c_{(2,1)}^{(0,m_2+1,m_1+m_2+1)}(k_1+1, k_2+1, k_3) - (1-q^{m_2})c_{(2,1)}^{(0,m_2,m_1+m_2)}(k_1, k_2, k_3).$$

Proposition 2.10. *Let δ_j be as in Proposition 2.8 and assume that $\lambda_{n-i} = \lambda_{n-i+1}$ for some i . Then:*

$$c_{(a_{n-s+1}, \dots, a_{n-s+i}, a_{n-s+i+1}, \dots, a_n)}^{(0, \dots, 0, \lambda_{n-s+1}+1, \dots, \lambda_n+1)}(k_1 + \delta_1, \dots, k_n + \delta_n) = c_{(a_{n-s+1}, \dots, a_{n-s+i+1}, a_{n-s+i}, \dots, a_n)}^{(0, \dots, 0, \lambda_{n-s+1}+1, \dots, \lambda_n+1)}(k_1 + \delta_1, \dots, k_n + \delta_n).$$

Proof. We prove proposition by induction on the number of cells in λ using Proposition 2.8.

Assume that the proposition is proved for all partitions λ' such that $|\lambda'| < |\lambda|$. For $\mathbf{a} = (a_{n-s+1}, \dots, a_{n-i}, a_{n-i+1}, \dots, a_n)$ we define

$$t_{n-i}\mathbf{a} = (a_{n-s+1}, \dots, a_{n-i+1}, a_{n-i}, \dots, a_n).$$

For a subset $B \subset \{1, \dots, n\}$, $|B| = s' \leq s$ we define

$$\mathbf{a}(B) = (b_{n-s'+1}, \dots, b_n), \quad (t_{n-i}\mathbf{a})(B) = (\tilde{b}_{n-s'+1}, \dots, \tilde{b}_n)$$

Using Proposition 2.8 we have:

$$\begin{aligned} c_{\mathbf{a}}^{(0, \dots, 0, \lambda_{n-s+1}+1, \dots, \lambda_n+1)}(k_1 + \delta_1, \dots, k_n + \delta_n) &= \sum_{B: |B|=s'} c_{\mathbf{a}(B)}^{\lambda}(k_1, \dots, k_n) q^{\sum_{j: b_j < a_j} \lambda_j}; \\ c_{t_{n-i}\mathbf{a}}^{(0, \dots, 0, \lambda_{n-s+1}+1, \dots, \lambda_n+1)}(k_1 + \delta_1, \dots, k_n + \delta_n) &= \sum_{B: |B|=s'} c_{(t_{n-i}\mathbf{a})(B)}^{\lambda}(k_1, \dots, k_n) q^{\sum_{j: \tilde{b}_j < (t_{n-i}\mathbf{a})_j} \lambda_j}. \end{aligned}$$

Our goal is to prove that

$$c_{(t_{n-i}\mathbf{a})(B)}^{\lambda}(k_1, \dots, k_n) q^{\sum_{j: b_j < a_j} \lambda_j} = c_{\mathbf{a}(B)}^{\lambda}(k_1, \dots, k_n) q^{\sum_{j: \tilde{b}_j < (t_{n-i}\mathbf{a})_j} \lambda_j}$$

for any B . More precisely, we will prove that the powers of q are equal and that

$$(2.13) \quad c_{(t_{n-i}\mathbf{a})(B)}^{\lambda}(k_1, \dots, k_n) = c_{\mathbf{a}(B)}^{\lambda}(k_1, \dots, k_n).$$

We claim that there are two possibilities: either $(t_{n-i}\mathbf{a})(B) = \mathbf{a}(B)$ or $(t_{n-i}\mathbf{a})(B) = (t_{n-i}(\mathbf{a}(B)))$ (we note that this claim implies the equality (2.13) by induction on $|\lambda|$). First, we note that $b_j = \tilde{b}_j$ for $j \geq n-i+2$. We denote by B' the set $B \setminus \{b_{n-i+2}, \dots, b_n\}$. To prove the desired claim, we consider two cases. First, assume that either

$$(2.14) \quad b_j < \min(a_{n-i}, a_{n-i+1}) \text{ or } b_j \geq \max(a_{n-i}, a_{n-i+1}) \text{ for all } j$$

or

$$(2.15) \quad \min(a_{n-i}, a_{n-i+1}) \leq b_j < \max(a_{n-i}, a_{n-i+1}) \text{ for all } j.$$

Then $(t_{n-i}\mathbf{a})(B) = \mathbf{a}(B)$ thanks to Proposition 2.4. Moreover, $\sum_{j: b_j < a_j} \lambda_j = \sum_{j: \tilde{b}_j < (t_{n-i}\mathbf{a})_j} \lambda_j$. In fact, we know that $\lambda_{n-i} = \lambda_{n-i+1}$ and one of the conditions (2.14), (2.15) is satisfied. Hence the sums above are equal.

Now assume that both conditions (2.14), (2.15) are not satisfied. Then $(t_{n-i}\mathbf{a})(B) = (t_{n-i}(\mathbf{a}(B)))$ because of Proposition 2.4. In particular, $\sum_{j: b_j < a_j} \lambda_j = \sum_{j: \tilde{b}_j < (t_{n-i}\mathbf{a})_j} \lambda_j$ because the summands of the sums coincide. \square

Proposition 2.11. *For antidominant λ one has*

$$\text{ch}_{KM} W_\lambda(x_1, \dots, x_n, q) = \sum c_{(s, s-1, \dots, 1)}^\lambda(k_1+1, \dots, k_s+1, k_{s+1}, \dots, k_n) x_1^{k_1} \dots x_n^{k_n}.$$

Proof. We know (see [S]) that $\text{ch}_{KM} W_\lambda(x_1, \dots, x_n, q) = E_\lambda(x; q, 0)$. Now we compute $E_\lambda(x; q, 0)$ using the Haglund-Haiman-Loehr formula. One has

$$E_\lambda(x; q, 0) = \sum_{\sigma \text{ non-attacking}} x^\sigma q^{\text{maj}(\sigma)} 0^{\text{coinv}(\hat{\sigma})}.$$

We note that $\text{coinv}(\sigma) \geq 0$. Indeed,

$$\text{coinv}(\hat{\sigma}) = \sum_{u \in dg'} a(u) - |\text{Inv}'(\hat{\sigma})| + \sum_{u \in \text{Des}(\hat{\sigma})} a(u).$$

For any two boxes $u, u' \in \text{leg}(u)$ we have that if $\sigma(u) > \sigma(u')$ and $\sigma(u') > \sigma(d(u))$ then $\sigma(u) > \sigma(d(u))$, so similarly to the proof of Proposition 2.4 we have that $\text{coinv}(\hat{\sigma}) \geq 0$ and $\text{coinv}(\hat{\sigma}) = 0$ if and only if σ is obtained by following inverse rule of filling:

Assume that we have filled the i -th row. Let S be the set of elements of the $(i+1)$ -st row. We fill the $(i+1)$ -st row of the diagram from right to left. If S' is the set of elements of S that are not used before, then into the cell v we put:

- (i) $\max\{a \in S', a \leq \hat{\sigma}(d(v))\}$, if $\{a \in S', a \leq \hat{\sigma}(d(v))\} \neq \emptyset$;
- (ii) $\max\{a \in S'\}$, if $\{a \in S', a \geq \hat{\sigma}(d(v))\} = \emptyset$.

We conclude that

$$E_\lambda(x; q, 0) = \sum c_{(s, s-1, \dots, 1)}^\lambda(k_1+1, \dots, k_s+1, k_{s+1}, \dots, k_n) x_1^{k_1} \dots x_n^{k_n}.$$

Hence

$$\sum c_{(s, s-1, \dots, 1)}^\lambda(k_1+1, \dots, k_s+1, k_{s+1}, \dots, k_n) x_1^{k_1} \dots x_n^{k_n} = \text{ch}_{KM} W_\lambda(x_1, \dots, x_n, q).$$

But $\text{ch}_{KM} W_\lambda(x_1, \dots, x_n, q)$ is a symmetric function in x_i 's. Therefore we obtain:

$$\sum c_{(s, s-1, \dots, 1)}^\lambda(k_1+1, \dots, k_s+1, k_{s+1}, \dots, k_n) x_1^{k_1} \dots x_n^{k_n} = \text{ch}_{KM} W_\lambda(x_1, \dots, x_n, q).$$

\square

2.4. Scrolled case. Recall the mappings

$$\pi(\lambda_1, \dots, \lambda_n) = (\lambda_n + 1, \lambda_1, \dots, \lambda_{n-1});$$

$$\Psi f(x_1, \dots, x_n) = x_1 f(x_2, \dots, x_n, q^{-1}x_1).$$

The Knop-Sahi recurrence states that

$$(2.16) \quad E_{\pi(\mu)}(x; q, t) = q^{\mu_n} \Psi E_\mu(x; q, t).$$

Proposition 2.12. *Consider a partition $\lambda = (\lambda_1, \dots, \lambda_n)$, $\lambda_1 \leq \dots \leq \lambda_n$, $\lambda_1 = \dots = \lambda_{n-s} = 0 \neq \lambda_{n-s+1}$. Then*

$$E_{\pi^r \lambda}(x; q^{-1}, \infty) = \sum_{k_1, \dots, k_n \geq 0} x_1^{k_1+1} \dots x_r^{k_r+1} x_{r+1}^{k_{r+1}} \dots x_n^{k_n} \times \\ c_{(n-s+r+1, \dots, n, 1, \dots, r)}^{(0, \dots, 0, \lambda_{n-s+1}+1, \dots, \lambda_n+1)}(k_1+1, \dots, k_r+1, k_{r+1}, \dots, k_{n-s+r}, k_{n-s+r+1}+1, \dots, k_n+1).$$

Proof. It is an immediate consequence of Proposition 2.4 for partition $\pi^r(\lambda)$. \square

Lemma 2.13.

$$q^{\lambda_{n-r+1} + \dots + \lambda_n - k_{n-r+1} - \dots - k_n} \times \\ c_{(n-s+r+1, \dots, n, 1, \dots, r)}^{0, \dots, 0, \lambda_{n-s+1}+1, \dots, \lambda_n+1}(k_{n-r+1}+1, \dots, k_n+1, k_1+1, \dots, k_r+1, k_{r+1}, \dots, k_{n-s+r}+1) = \\ c_{(n-s+1, \dots, n)}^{0, \dots, 0, \lambda_{n-s+1}+1, \dots, \lambda_n+1}(k_1+1, \dots, k_r+1, k_{r+1}, \dots, k_{n-s+r}, k_{n-s+r+1}+1, \dots, k_n+1)$$

Proof. Immediate consequence of equation (2.16) and Proposition 2.12. \square

Example 2.14. Consider the case $n = 3$ and partition $\lambda = (0, m_2, m_1 + m_2)$. Then by Proposition 2.12 we have:

$$E_{(m_1+m_2+1, 0, m_2)}(x_1, x_2, x_3; q^{-1}, \infty) = x_1 \sum_{k_1, k_2, k_3 \geq 0} c_{(3,1)}^\lambda(k_1, k_2, k_3) x_1^{k_1} x_2^{k_2} x_3^{k_3}.$$

Using equation (2.16) we obtain

$$c_{(2,3)}^\lambda(k_1, k_2 + 1, k_3 + 1) = q^{m_1+m_2-k_3} c_{(3,1)}^\lambda(k_3 + 1, k_1, k_2 + 1).$$

3. THE CHEREDNIK-ORR CONJECTURE

Let $\mathfrak{g} = \mathfrak{sl}_n$. If $\lambda = \sum_{i=1}^{n-1} m_i \omega_i$, then the corresponding diagram is equal to $\lambda = (\lambda_1, \dots, \lambda_n)$, where $\lambda_i = m_i + \dots + m_{n-1}$ ($\lambda_n = 0$). The diagram of the antidominant weight λ^* is reversed, i.e. is given by $(\lambda_n, \dots, \lambda_1)$. For example, the diagram corresponding to $(m\omega_r)^*$ is equal to $(\underbrace{0, \dots, 0}_{n-r}, m, \dots, m)$.

3.1. Rectangular diagrams. In this subsection we use Proposition 1.4. Assume that the diagram $dg'(\lambda)$ is rectangular of length r and height m . By Proposition 2.4 we know that an appropriate filling is completely determined by m subsets of r elements from $1, \dots, n$. We know that an order of elements in i -th row determines the order of elements in $(i+1)$ -st row. Because of Proposition 2.10 the order of elements a_{n-r+1}, \dots, a_n in

$$(3.1) \quad c_{(a_{n-r+1}, \dots, a_n)}^{(0, \dots, 0, m+1, \dots, m+1)}(k_1 + \delta_1, \dots, k_n + \delta_n)$$

(δ_i as in Proposition 2.8) is not important, we only care about the set $\{a_{n-r+1}, \dots, a_n\}$. We use Proposition 2.8 to write recurrent equations for (3.1).

Example 3.1. Let $r = 2$, $i < j$. Then

$$\begin{aligned} c_{\{i,j\}}^\lambda(k_1, \dots, k_{i-1}, k_i + 1, k_{i+1}, \dots, k_{j-1}, k_j + 1, k_{j+1}, \dots, k_n) = \\ q^{2m} \sum_{p,q < i} c_{\{p,q\}}^\lambda(k_1, \dots, k_n) + q^m \sum_{p < i, q \geq i} c_{\{p,q\}}^\lambda(k_1, \dots, k_n) + \\ q^m \sum_{p \geq i, q < j} c_{\{p,q\}}^\lambda(k_1, \dots, k_n) + \sum_{p \geq i, q \geq j} c_{\{p,q\}}^\lambda(k_1, \dots, k_n) \end{aligned}$$

Note that the most important case for us is $c_{\{n-r+1, \dots, n\}}^\lambda$. Using Proposition 2.12 we have:

$$(3.2) \quad E_{\pi^r(\lambda)}(x, q^{-1}, \text{inf ty}) = x_1 \dots x_r \sum_{k_i \geq 0} c_{\{1, \dots, r\}}^\lambda(k_1 + 1, \dots, k_r + 1, k_{r+1}, \dots, k_n) x_1^{k_1} \dots x_n^{k_n}.$$

We note that $\pi^r(\lambda) = (m+1, \dots, m+1, 0, \dots, 0)$.

Theorem 3.2. Let $\lambda = (m\omega_r)^*$. Then we have:

$$E_\lambda(x; q^{-1}, \infty) = \text{ch}_{PBW} W_\lambda^{gr}(x; q, q).$$

Proof. Using Proposition 2.11 we have:

$$\text{ch}_{KM} W_\lambda(x_1, \dots, x_n, q) = \sum c_{\{1, 2, \dots, r\}}^\lambda(k_1 + 1, \dots, k_r + 1, k_{r+1}, \dots, k_n) x_1^{k_1} \dots x_n^{k_n}.$$

Now using Lemma 2.13 we obtain:

$$\begin{aligned} q^{rm - k_{n-r+1} - \dots - k_n} c_{\{1, 2, \dots, r\}}^\lambda(k_1 + 1, \dots, k_r + 1, k_{r+1}, \dots, k_n) = \\ c_{\{n-r+1, \dots, n\}}^\lambda(k_1, \dots, k_{n-r}, k_{n-r+1} + 1, \dots, k_n + 1). \end{aligned}$$

So we have:

$$\begin{aligned}
E_\lambda(x; q^{-1}, \infty) &= \\
&= \sum c_{\{n-r+1, \dots, n\}}^\lambda(k_1, \dots, k_{n-r}, k_{n-r+1} + 1, \dots, k_n + 1) x_1^{k_1} \dots x_n^{k_n} = \\
&= \sum q^{rm - k_{n-r+1} - \dots - k_n} c_{\{1, 2, \dots, r\}}^\lambda(k_1 + 1, \dots, k_r + 1, k_{r+1}, \dots, k_n) x_1^{k_1} \dots x_n^{k_n} = \\
&= \sum c_{\{1, 2, \dots, r\}}^\lambda(k_1 + 1, \dots, k_r + 1, k_{r+1}, \dots, k_n) (qx_1)^{k_1} \dots (qx_{n-r})^{k_{n-r}} x_{n-r+1}^{k_{n-r+1}} \dots x_n^{k_n} = \\
&= \text{ch}_{KM} W_\lambda(qx_1, \dots, qx_{n-r}, x_{n-r+1}, \dots, x_n, q).
\end{aligned}$$

Then using Proposition 1.4 we complete the proof of this Theorem. \square

3.2. $m_1\omega_1 + m_2\omega_{n-1}$ -case. Let us fix the highest weight $m_1\omega_1 + m_2\omega_{n-1}$ (for $n = 3$ this is the general case). Then the corresponding diagram is of the form $(0, m_2, \dots, m_2, m_1 + m_2)$. Let $A = (a_1, \dots, a_{n-1})$ be a string of $n - 1$ different elements from the set $\{1, \dots, n\}$, $\{1, \dots, n\} \setminus \{a_1, \dots, a_{n-1}\} = \{\tilde{a}\}$. Let t_i be the transposition of i -th and $(i + 1)$ -st elements. Then Proposition 2.10 tells us that

$$\begin{aligned}
c_A^\lambda(k_1 + 1, \dots, k_{\tilde{a}-1} + 1, k_{\tilde{a}}, k_{\tilde{a}+1} + 1, \dots, k_n + 1) &= \\
c_{t_i A}^\lambda(k_1 + 1, \dots, k_{\tilde{a}-1} + 1, k_{\tilde{a}}, k_{\tilde{a}+1} + 1, \dots, k_n + 1),
\end{aligned}$$

for $1 \leq i \leq n - 3$. So the only essential parameters are \tilde{a} and a_{n-1} (we can not interchange a_{n-1} with a_{n-2} , because $\lambda_n > \lambda_{n-1}$). We denote these polynomials by $c_{\tilde{a}|a_{n-1}}^\lambda(k_1, \dots, k_n)$.

Proposition 3.3. *Assume that $\lambda^* = m_1\omega_1 + m_2\omega_{n-1}$. Then:*

$$\begin{aligned}
\text{ch}_{KM} W_\lambda(x, q) &= \\
&= \sum_{k_i \geq 0} \sum_{\substack{p_i + \sum_{\alpha \neq i} l_\alpha = k_i, \\ l_1 + \dots + l_n = m_2, \\ p_1 + \dots + p_n = m_1}} q^{l_1 p_1} \binom{m_2}{l_1, \dots, l_n}_q \binom{m_1}{p_1, \dots, p_n}_q x_1^{k_1} \dots x_n^{k_n}.
\end{aligned}$$

$$\begin{aligned}
\text{ch}_{PBW} W_\lambda^{gr}(x, q, q) &= \\
&= \sum_{k_i \geq 0} \sum_{\substack{p_i + \sum_{\alpha \neq i} l_\alpha = k_i, \\ l_1 + \dots + l_n = m_2, \\ p_1 + \dots + p_n = m_1}} q^{l_1 p_1 + l_2 + \dots + l_n + p_1 + \dots + p_{n-1}} \binom{m_2}{l_1, \dots, l_n}_q \binom{m_1}{p_1, \dots, p_n}_q x_1^{k_1} \dots x_n^{k_n}.
\end{aligned}$$

Proof. We construct a bijection η between elements of the PBW-basis for W_λ and pairs of strings: one of them of the length m_1 and other of the length m_2 , both filled by elements of the set $\{1, \dots, n\}$. We denote the pairs of filled strings by $\tau = a_1, \dots, a_{m_2} | b_1, \dots, b_{m_1}$. Put

$$\eta(\tau) = \prod_{a_i = n} e_{1, n-1} \otimes t^{\{j < i | a_j = 1\}} \prod_{i | a_i \neq 1, n} e_{1, a_i - 1} \otimes t^{\{j < i | a_j < a_i \text{ or } a_j = n\}}$$

$$\prod_{b_i=1} e_{1,n-1} \otimes t^{|\{j < i | a_j = n\}| + |\{a_j = 1\}|} \prod_{i | b_i \neq 1, n} e_{b_i-1, n-1} \otimes t^{|\{j < i | b_j < b_i \text{ or } b_j = n\}|}.$$

Note that for any τ $\eta(\tau)$ satisfies equalities (1.2), (1.3), (1.4) and comparing the numbers of elements in both sets we obtain that η is indeed a bijection. Put

$$d(\tau) = |\{(i < j) | a_i < a_j < n \text{ or } a_i = n, a_j < n\}| + |\{(i < j) | b_i < b_j < n \text{ or } b_i = n, b_j < n\}| + |\{i | a_i = 1, b_i = 1\}|.$$

Then by definition of η we have $d(\eta(\tau)) = d(\tau)$, where d in the right hand side is the Kac-Moody energy operator.

Note that

$$\deg_{PBW}(\eta(\tau)) = |\{i | a_i \neq 1\}| + |\{i | b_i \neq n\}|$$

and its weight is (k_1, \dots, k_n) , where $k_i = p_i + \sum_{\alpha \neq i} l_\alpha$, and $p_i = |\{j | b_j = i\}|$, $l_i = |\{j | a_j = i\}|$.

Fix l_i and p_i . Then sum of $q^{d(\tau)}$ for such elements is $q^{l_1 p_1} \binom{m_2}{l_1, \dots, l_n}_q \binom{m_1}{p_1, \dots, p_n}_q$. Indeed, the last summand in the definition of $d(\tau)$ is $l_1 p_1$ and by definition of the q -binomial coefficients:

$$\begin{aligned} \binom{m_2}{l_1, \dots, l_n}_q &= \sum q^{|\{i < j | a_i < a_j < n \text{ or } a_i = n, a_j < n\}|} \\ \binom{m_1}{p_1, \dots, p_n}_q &= \sum q^{|\{i < j | b_i < b_j < n \text{ or } b_i = n, b_j < n\}|} \end{aligned}$$

Similarly we obtain second equation of the Proposition. This completes the proof of the Lemma. \square

Theorem 3.4. *The Cherednik-Orr conjecture is true for $\lambda^* = m_1 \omega_1 + m_2 \omega_{n-1}$*

Proof. Using Proposition 2.11 and Theorem 1.8 we have:

$$\begin{aligned} c_{n|1}^{(0, m_2+1, \dots, m_2+1, m_2+m_1+1)}(k_1+1, \dots, k_{n-1}+1, k_n) &= \\ &= \sum_{\substack{p_i + \sum_{\alpha \neq i} l_\alpha = k_i \\ l_1 + \dots + l_n = m_2, \\ p_1 + \dots + p_n = m_1}} q^{l_1 p_1} \binom{m_2}{l_1, \dots, l_n}_q \binom{m_1}{p_1, \dots, p_n}_q. \end{aligned}$$

Using Proposition 2.8 we have for $j \geq 2$:

$$\begin{aligned} c_{j|1}^{(0, m_2+1, \dots, m_2+1, m_2+m_1+1)}(k_1+1, \dots, k_{j-1}+1, k_j, k_{j+1}+1, \dots, k_n+1) &= \\ &= \sum_{i=2}^j c_{i|1}^\lambda(k_1, \dots, k_n) + \sum_{i=j+1}^n c_{i|1}^\lambda(k_1, \dots, k_n) q^{m_2} + c_{1|1}^\lambda(k_1, \dots, k_n). \end{aligned}$$

Subtracting the equations for consequent j 's we obtain:

$$c_{j|1}^{(0,m_2+1,\dots,m_2+1,m_2+m_1+1)}(k_1+1,\dots,k_{j-1}+1,k_j,k_{j+1}+1,\dots,k_n+1) = \\ c_{j+1|1}^{(0,m_2+1,\dots,m_2+1,m_2+m_1+1)}(k_1+1,\dots,k_j+1,k_{j+1},k_{j+2}+1,\dots,k_n+1) - \\ (1-q^{m_2})c_{j+1|1}^\lambda(k_1,\dots,k_n).$$

We claim that

$$c_{j|1}^{(0,m_2+1,\dots,m_2+1,m_2+m_1+1)}(k_1+1,\dots,k_{j-1}+1,k_j,k_{j+1}+1,\dots,k_n+1) = \\ \sum_{\substack{p_i+l_{i+1}+\dots+l_n+l_1+\dots+l_{i-1}=k_i, \\ l_1+\dots+l_n=m_2, \\ p_1+\dots+p_n=m_1}} q^{l_1p_1+l_{j+1}+\dots+l_n} \binom{m_2}{l_1,\dots,l_n}_q \binom{m_1}{p_1,\dots,p_n}_q.$$

Indeed, we know that it is true for $j = n$ and

$$\sum_{\substack{p_i+\sum_{\alpha \neq i} l_\alpha = k_i, \\ l_1+\dots+l_n=m_2, \\ p_1+\dots+p_n=m_1}} q^{l_1p_1+l_{j+1}+\dots+l_n} \binom{m_2}{l_1,\dots,l_n}_q \binom{m_1}{p_1,\dots,p_n}_q - \\ (1-q^{m_2}) \sum_{\substack{p_i+\sum_{\alpha \neq i} l_\alpha = k_i-1, i \neq j, \\ p_j+\sum_{\alpha \neq j} l_\alpha = k_j, \\ l_1+\dots+l_n=m_2-1, \\ p_1+\dots+p_n=m_1}} q^{l_1p_1+l_{j+1}+\dots+l_n} \binom{m_2}{l_1,\dots,l_n}_q \binom{m_1}{p_1,\dots,p_n}_q = \\ \sum_{\substack{p_i+\sum_{\alpha \neq i} l_\alpha = k_i, \\ l_1+\dots+l_n=m_2, \\ p_1+\dots+p_n=m_1}} \left(q^{l_1p_1+l_{j+1}+\dots+l_n} \binom{m_2}{l_1,\dots,l_n}_q \binom{m_1}{p_1,\dots,p_n}_q - \right. \\ \left. (1-q^{m_2}) q^{l_1p_1+l_{j+1}+\dots+l_n} \binom{m_2-1}{l_1,\dots,l_{j-1},l_j-1,l_{j+1},\dots,l_n}_q \binom{m_1}{p_1,\dots,p_n}_q \right) = \\ \sum_{\substack{p_i+\sum_{\alpha \neq i} l_\alpha = k_i, \\ l_1+\dots+l_n=m_2, \\ p_1+\dots+p_n=m_1}} q^{l_1p_1+l_{j+1}+\dots+l_n+l_j} \binom{m_2}{l_1,\dots,l_n}_q \binom{m_1}{p_1,\dots,p_n}_q$$

In particular we obtain:

$$c_{2|1}^{(0,m_2+1,\dots,m_2+1,m_2+m_1+1)} = \\ \sum_{\substack{p_i+\sum_{\alpha \neq i} l_\alpha = k_i, \\ l_1+\dots+l_n=m_2, \\ p_1+\dots+p_n=m_1}} q^{l_1p_1+l_3+\dots+l_n} \binom{m_2}{l_1,\dots,l_n}_q \binom{m_1}{p_1,\dots,p_n}_q.$$

But using Lemma 2.13 we have:

$$\begin{aligned}
& c_{1|n}^{(0, m_2+1, \dots, m_2+1, m_2+m_1+1)}(k_1, k_2+1, \dots, k_n+1) = \\
& q^{m_2+m_1-k_n} c_{2|1}^{(0, m_2+1, \dots, m_2+1, m_2+m_1+1)}(k_n+1, k_1, k_2+1, \dots, k_{n-1}+1) = \\
& \sum_{\substack{p_i + \sum_{\alpha \neq i} l_\alpha = k_{i-1} \pmod{n}, \\ l_1 + \dots + l_n = m_2, \\ p_1 + \dots + p_n = m_1}} q^{l_1 p_1 + l_3 + \dots + l_n + p_2 + \dots + p_n + l_1} \binom{m_2}{l_1, \dots, l_n}_q \binom{m_1}{p_1, \dots, p_n}_q.
\end{aligned}$$

We thus obtain exactly the coefficient of $x_1^{k_1} \dots x_n^{k_n}$ in the formula for PBW-character of Proposition 3.3. \square

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